AD 749113

TELECOMMUNICATIONS Technical Memorandum

Scattering by Water Waves Generated by a Moving Pressure Point

R.H. Ott

G.A. Hufford

A.Q. Howard, Jr.

BOULDER COLORADO

22

OFFICE OF TELECOMMUNICATIONS

INSTITUTE FOR TELECOMMUNICATION SCIENCES

IMPORTANT NOTICE

Technical Memorandums are used to insure prompt distribution of studies which, though of interest to the sponsor and/or a limited readership, may not be ready or appropriate for formal publication. Since these papers may later be published in a modified form to include more recent information or research results, abstracting, citing, or reproducing this paper in the open literature is not encouraged. Contact the author for additional information on the subject matter discussed in the Memorandum.

OFFICE OF TELECOMMUNICATIONS

BOULDER COLORADO

U.S. DEPARTMENT OF COMMERCE
OFFICE OF TELECOMMUNICATIONS
Institute for Telecommunication Sciences

SCATTERING BY WATER WAVES GENERATED BY A MOVING PRESSURE POINT

R. H. Ott, G. A. Hufford, and A. Q. Howard, Jr.

Sponsor: This work was supported by the
Advanced Research Projects Agency,
ARPA Order Number 1809

Contract Monitor: Office of Naval Research,
Code 418
Contract No. NAonr-23-71

Institute for Telecommunication Sciences
Boulder, Colorado
August 1972

CONTENTS

Preceding page blank

SCATTERING BY WATER WAVES GENERATED BY A MOVING PRESSURE POINT

R. H. Ott, G. A. Hufford and A. Q. Howard, Jr.

ABSTRACT

An analysis for scattering of scalar waves in air by a wake on the surface of water is developed. The wake on the water surface is generated as a concentrated pressure point travels over the free water surface. The wake has characteristics resembling a diffraction grating. The grating produces maximum constructive interference (resonance) in the backscattered wave under certain conditions. At resonance the acoustic wavelength, water wavelength and angle of incident satisfy a simple geometrical condition.

1. INTRODUCTION

An analysis for studying the scatter of scalar waves in air by a wake on the surface of water is developed. The wake on the water surface is generated by a concentrated pressure point traveling with uniform velocity over deep water. The only surface force is pressure, the only body force is gravity and the motion is irrotational.

A number of authors have investigated the wake left by a pressure point as it travels over the water. The wake is largely confined to the sector bounded by two cuspidal lines which form angles of $\sin^{-1}(\frac{1}{3})$ with the track of the pressure point. The disturbance drops off exponentially outside the cuspidal lines. This angle was first determined

by Lord Kelvin (1887). Within this sector the wake consists of a system of waves that envelope the hull lengthwise and is interwoven with a system of transverse waves. The amplitude of the surface within this sector has been treated by a number of authors (e.g., Peters (1949), Ursell (1960), Hufford (1972)).

In this paper we introduce the boundary value problem for the velocity potential $\hat{\varphi}(x, y, z)$ and the surface elevation $\zeta(x, y)$. The velocity potential satisfies Laplace's equation. The velocity potential and surface elevation together satisfy two boundary conditions: 1) the pressure equation and 2) the equation of continuity across the free surface.

Sommerfeld (1950) is a good introduction to the theory of waves and the equations of hydrodynamics applied to our problem.

The approach to the solution to Laplace's equation plus boundary conditions is through the two-dimensional Fourier transform (Havelock (1918), Hogner (1923), Peters (1949)). These equations are solved in terms of the transformed potential $\hat{\Phi}(\xi, \eta, z)$ and surface elevation $Z(\xi, \eta)$. The final step consists of identifying the transform $Z(\xi, \eta)$ as the solution to our scattering problem. A simple expression for the scattered field is obtained which depends upon the acoustic wave number k, the water wave number k_w and the direction of the source and observer relative to the track of the pressure point. This expression has characteristics resembling the field scattered by a diffraction grating. The interference phenomena in this problem are explained physically by reflections from neighbouring curves having water wave phase differences which are multiples of 2π.

2. ANALYSIS

Consider a monochromatic scalar wave having $e^{i\omega t}$ time dependence incident on the surface $z = \zeta(x, y)$ in figure 1. Assume the potential φ satisfies a sound-hard boundary condition (corresponding to a vertically polarized incident wave in the electromagnetic problem)

$$\frac{\partial \varphi}{\partial n} = 0 , \quad \varphi \text{ on } \zeta \tag{1}$$

where differentiation is with respect to the outward normal. The scattered potential from Helmholtz's formula is

$$\varphi^s(P) = \frac{1}{4\pi} \int_\zeta \varphi(Q) \frac{\partial G}{\partial n} \, da \tag{2}$$

where P is the location of the source and observer for the case of backscatter and Q is the integration point and ζ is the surface of integration. The free-space Green's function is

$$G = \frac{e^{-ikr_2}}{r_2} \tag{3}$$

with

$$r_2 = PQ \quad . \tag{4}$$

Neglecting amplitude terms of the order $(kr_2)^{-1}$, we obtain from (3)

$$\frac{\partial G}{\partial n} \simeq -ikG \frac{\partial r_2}{\partial n} \tag{5}$$

We assume that near grazing the field at the integration point Q is the unperturbed field of a point source at P radiating over a perfectly conducting flat surface; i.e.,

$$\varphi(Q) = 2\varphi^{i}(Q) = 2\,\frac{e^{-ikr_2}}{r_2} \tag{6}$$

The elemental area da projected onto the x-y plane becomes

$$da = \frac{dx\,dy}{|\underline{e}_n \cdot \underline{e}_z|} \tag{7}$$

where $\underline{e}_n$ is the outward directed normal to ζ in figure 1 and $\underline{e}_z$ is a unit vector along the positive z-axis. The normal derivative of r_2 in (5) is

$$\frac{\partial r_2}{\partial n} = \underline{e}_{r_2} \cdot \underline{e}_n \tag{8}$$

with $\underline{e}_{r_2}$ a unit vector from the observation point P to the integration point Q. The unit normal is

$$\underline{e}_n = \left(-\underline{\nabla}_T \zeta + \underline{e}_z\right) |\underline{e}_n \cdot \underline{e}_z| \tag{9}$$

with

$$\underline{\nabla}_T \zeta = \underline{e}_x \frac{\partial \zeta}{\partial x} + \underline{e}_y \frac{\partial \zeta}{\partial y} \tag{10}$$

and $\underline{\nabla}_T\zeta$ is shown in plan view in Fig. 1. That is, $\underline{\nabla}_T\zeta$ is the projection (to within the normalization $\underline{e}_n \cdot \underline{e}_z$) of $\underline{e}_n$ onto the x-y plane. If the surface elevation ζ is small compared with the acoustic wavelength λ and the slopes $\partial\zeta/\partial x$ and $\partial\zeta/\partial y$ are small compared to unity we find for an observer close to the x-y plane

$$\underline{e}_k \cdot \underline{e}_z \cong 0 \tag{11}$$

with $\underline{e}_k \cong \underline{e}_{r_2}$ a unit vector from P to the origin in figure 1. From (8), (9), (10) and (11)

$$\frac{\partial r_2}{\partial n} \cong \left(-\underline{e}_k \cdot \underline{\nabla}_T\zeta\right) |\underline{e}_n \cdot \underline{e}_z| \tag{12}$$

Substituting (3), (5), (6), (7) and (12) into (2) gives

$$\varphi^s(P) = \frac{ik}{2\pi} \int_{\zeta'} \left(\underline{e}_k \cdot \underline{\nabla}_T \zeta \right) \frac{e^{-i2kr_2}}{r_2^2} \, dx\, dy \tag{13}$$

where ζ' is the projection of ζ onto the x-y plane.

From figure 1.

$$r_2 \cong R + \underline{e}_k \cdot \underline{r} \tag{14}$$

or

$$r_2 \cong R + (x \cos\theta - y \sin\theta) \tag{15}$$

Substituting (15) into (13) and making the usual approximation regarding the amplitude factor r_2^{-1} we obtain

$$\varphi^s(P) = \frac{ik e^{-i2kR}}{2\pi R^2} \int_{\zeta'} \left(\underline{e}_k \cdot \underline{\nabla}_T \zeta \right) e^{-i2k(x\cos\theta - y\sin\theta)} \, dx\, dy \tag{16}$$

which is a two-dimensional Fourier transform of $\left(\underline{e}_k \cdot \underline{\nabla}_T \zeta \right)$. The surface ζ' is assumed to extend from $-\infty$ to $+\infty$ along the x- and y-axes.

From

$$\underline{e}_k \cdot \underline{\nabla}_T \zeta = \cos\theta \frac{\partial \zeta}{\partial x} - \sin\theta \frac{\partial \zeta}{\partial y} \tag{17}$$

and the property for the Fourier transform of the derivative of a function, we obtain

$$\varphi^s(P) = \frac{-k^2}{\pi R^2} e^{-i2kR} Z(\xi, \eta) \tag{18}$$

where

$$Z(\xi,\eta) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \zeta(x,y)\, e^{-i(\xi x+\eta y)} \tag{19}$$

is the Fourier transform of the surface elevation $\zeta(x,y)$. We have assumed in (16) that ζ' is infinite in extent and at the same time to arrive at (16) we make approximations regarding the phase kr_2 in (14) and the amplitude factor r_2^{-1}. These approximations can be made rigorous using the theory of generalized functions (Hufford, 1972). Hufford shows, by considering a sequence of surfaces $\{\zeta n\}$ whose limit is ζ, that in the limit as $\zeta_n \to \infty$ the scattered field approaches the result in (16).

The variables ξ and η in (19) become

$$\begin{aligned} \xi &= 2k\cos\theta \\ \eta &= -2k\sin\theta \end{aligned} \tag{20}$$

when they are used to denote the components of the acoustic wave number; otherwise, they will be thought of as the components of the water wave number.

The result in (18) shows that finding the field scattered by a rather complicated surface ζ is reduced to the problem of finding the Fourier transform of ζ. In order to find the Fourier transform of ζ we need to introduce the hydrodynamical equations for a pressure point moving at constant velocity across the water surface. The hydrodynamical problem is classical (e.g., Lamb, 1879) and has been treated by a number of authors. Most of the classical treatments are concerned with the form of ζ. However, here we are only interested in its transform Z which is somewhat easier to derive.

We assume the pressure point moves with velocity v in the negative x direction and at the instant of observation it is located at the origin. Assuming irrotational flow we may introduce a velocity potential $\hat{\varphi}$ and the equations of hydrodynamics become: the equation of incompressibility

$$\nabla^2 \hat{\varphi} = 0 \quad , \quad z < 0 \tag{21}$$

the equation of motion

$$v \frac{\partial \hat{\varphi}}{\partial x} = g \zeta + \frac{1}{\rho} P(x, y) \quad , \quad z = 0 \tag{22}$$

and the condition that the normal component of velocity be zero at a fixed boundary

$$\frac{\partial \hat{\varphi}}{\partial z} + v \frac{\partial \zeta}{\partial x} = 0 \quad , \quad z = 0 \tag{23}$$

In these equations, ρ represents the density of the water, $P = P_o \delta(x, y)$ is the magnitude of the pressure and g the acceleration of gravity. It should be pointed out that (22) and (23) are linearized approximations and require $\hat{\varphi}$ and ζ be small. The origin of the xyz axes is moving in the negative x direction with a uniform velocity v. Now we take the two-dimensional Fourier transform over x and y; i.e.,

$$\hat{\Phi}(\xi, \eta, z) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \hat{\varphi}(x, y, z)\, e^{-i(\xi x + \eta y)} \tag{24}$$

together with the Fourier transform of ζ given in (19) and equations (21), (22) and (23) become

$$-(\xi^2 + \eta^2)\, \hat{\Phi} + \frac{\partial^2 \hat{\Phi}}{\partial z^2} = 0 \quad , \quad z < 0 \tag{25}$$

$$i \xi v \hat{\Phi} = g Z + \frac{P_o}{\rho} \quad , \quad z = 0 \tag{26}$$

$$\frac{\partial \hat{\Phi}}{\partial z} = -i v \xi Z \quad , \quad z = 0 \tag{27}$$

Eliminating Z from (26) and (27) gives

$$\frac{\xi^2 v^2}{g} \hat{\Phi} = \frac{\partial \hat{\Phi}}{\partial z} - \frac{i \xi v P_o}{g \rho} \quad , \qquad z = 0 \tag{28}$$

The solution of (25) is

$$\hat{\Phi} = \hat{\Phi}_0 \, e^{\sqrt{\xi^2 + \eta^2}\, z} \tag{29}$$

Substituting (29) into (28) and solving for $\hat{\Phi}_0$ gives

$$\hat{\Phi}_0 = \frac{-i v P_o}{g \rho} \frac{\xi}{\left[\frac{\xi^2 v^2}{g} - \sqrt{\xi^2 + \eta^2} \right]} \tag{30}$$

From (27) we can solve for Z

$$Z(\xi,\eta) = \frac{P_o}{g\rho} \frac{(\xi^2 + \eta^2)^{\frac{1}{2}}}{\left[\frac{\xi^2 v^2}{g} - (\xi^2 + \eta^2)^{\frac{1}{2}} \right]} \tag{31}$$

The water wave number is $k_w = g/v^2$ and substituting k_w into (31) yields the desired result

$$Z(\xi,\eta) = \left(\frac{P_o k_w}{g\rho} \right) \frac{(\xi^2 + \eta^2)^{\frac{1}{2}}}{\left[\xi^2 - k_w (\xi^2 + \eta^2)^{\frac{1}{2}} \right]} \tag{32}$$

the Fourier transform of $\zeta(x,y)$. The quantity $(P_o k_w / g\rho)$ has dimensions $(\text{length})^2$ and Z has dimensions $(\text{length})^3$. Substituting (32) into (18) and using the acoustical interpretation for ξ and η given in (20) gives

$$\varphi^s(P) = -\left(\frac{e^{-i2kR}}{\pi R^2}\right)\left(\frac{P_o k_w^2}{g\rho}\right)\frac{\left(k/k_w\right)^2}{\left[2\left(\frac{k}{k_w}\right)\cos^2\theta - 1\right]} \tag{33}$$

the scattered acoustic field. The scattered field has dimensions (length)$^{-1}$ which is the same as the incident field.

We now turn to an interpretation of (33). The denominator vanishes when

$$2\,k\cos^2\theta = k_w \tag{34}$$

The resulting singularity perhaps requires further analysis, but it is clear that it represents a "resonance". For the moment we will think of ξ and η as components of the water wave number and not the components of the acoustic wave number as defined in (20). This interpretation is motivated by noting that the denominator in (32) is identical to that in (33) and the result in (32) has nothing to do with the acoustic problem. It is the Fourier transform of the water surface.

The water wave phase is

$$\psi = x\xi + y\eta \tag{35}$$

and the stationary water phase points are solutions to

$$\frac{\partial\psi}{\partial\xi} = 0 \quad \left(\text{or } \frac{\partial\psi}{\partial\eta} = 0\right) \tag{36}$$

At resonance, from (32), ξ and η are related as

$$\xi^2 = k_w(\xi^2 + \eta^2)^{\frac{1}{2}} \tag{37}$$

Solving (35), (36) and (37) for x and y in terms of a parameter $u = \xi/k_w$ gives

$$x = a\,\frac{(2u^2-1)}{u^3} \tag{38}$$

$$y = \pm a\,\frac{(u^2-1)^{\frac{1}{2}}}{u^3} \tag{39}$$

with $a = \psi / k_w$. These two equations represent the surfaces of constant water wave phase. One water wave crest is shown in figure 2 where $a = \lambda_w$. To generate the next crest $a = 2\lambda_w$ and in general the p^{th} crest is generated by $a = p\lambda_w$. The cuspidal lines forming the boundary of the sector occupied by the wake may be determined from (38) and (39) by defining

$$\tan\alpha = \frac{y}{x} = \frac{(u^2-1)^{\frac{1}{2}}}{2u^2-1} \tag{40}$$

and finding the point where

$$\frac{\partial(\tan\alpha)}{\partial u} = 0 \tag{41}$$

Substituting (40) into (41) gives the point

$$u = \pm\sqrt{\frac{3}{2}} \tag{42}$$

Substituting (42) back into (40) gives the angular sector defining the wake (Kelvin, 1887)

$$\alpha_c = \pm \tan^{-1}\left(\frac{1}{2\sqrt{2}}\right) \cong \pm 19°28' \tag{43}$$

This cuspidal angle is shown in figure 2. For $1 \leq u \leq \sqrt{\frac{3}{2}}$, x and $|y|$ lie on the portion of the curve starting from the point $x = 1$, $y = 0$ and going to the cusp (transverse component). For $u \geq \sqrt{\frac{3}{2}}$, x and $|y|$ lie on the portion of the curve between the cusp and the origin (lengthwise component).

The tangent to the curve in figure 2 is given by

$$\frac{dy}{dx} = \frac{dy/du}{dx/du} = \frac{1}{\pm(u^2-1)^{\frac{1}{2}}} \tag{44}$$

The tangent to the cusps in figure 2 are given by (44) where $u^2 = 3/2$

$$\alpha_T = \tan^{-1}\sqrt{2} \cong \pm 54^\circ\, 44' \tag{45}$$

This angle is shown in figure 2. The angles α_c and α_T satisfy $\alpha_c = 2\alpha_T - \pi/2$.

Finally, we return to the interpretation of (33) from the scattering viewpoint. Consider our acoustic wave incident at an angle θ with respect to the negative x-axis as shown in figure 3. A tangent vector to the water wave phase surfaces in figure 3 (shown as a unit vector $\underline{e}_T$, in figure 3) is

$$\underline{T} = \underline{e}_x\left(\frac{-2u^2+3}{u^4}\right) + \underline{e}_y\left(\frac{-2u^2+3}{u^4\sqrt{u^2-1}}\right), \quad u \geq 1 \tag{46}$$

The incident acoustic wave will be perpendicular to the water wave phase surfaces giving maximum reflection in the backscatter direction (resonance) when

$$\underline{T} \cdot \underline{e}_k = 0 \tag{47}$$

with

$$\underline{e}_k = \underline{e}_x \cos\theta - \underline{e}_y \sin\theta \tag{48}$$

Substituting (46) and (48) into (47) gives

$$\tan\theta = (u^2-1)^{\frac{1}{2}} \tag{49}$$

It is not surprising that the result in (49) is the reciprocal of that in (44). The surfaces of constant water wave phase can be thought of as representing a diffraction grating. When twice the path length d (to

go in and out) is equal to the incident acoustic wave length λ, neighbouring water wave lines in the grating will interfere constructively producing maximum backscatter (resonance). Twice the distance d in figure 3 is

$$2d = 2\,\underline{R} \cdot \underline{e}_k \tag{50}$$

where

$$\underline{R} = x\underline{e}_x + y\underline{e}_y = \frac{\psi\lambda_w}{2\pi}\left\{\underline{e}_x \frac{(2u^2-1)}{u^3} + \underline{e}_y \frac{(u^2-1)^{\frac{1}{2}}}{u^3}\right\} \tag{51}$$

Substituting (48) and (51) into (50) yields

$$2d = \frac{\psi\lambda_w}{\pi}\left\{\cos\theta\,\frac{(2u^2-1)}{u^3} - \sin\theta\,\frac{(u^2-1)^{\frac{1}{2}}}{u^3}\right\} \tag{52}$$

This distance remains the same for any two adjacent water wave surfaces in the grating. Substituting for u in (52) in terms of θ from (49) gives

$$2d = \frac{\psi\lambda_w}{\pi}\cos^2\theta \tag{53}$$

The phase difference ψ between two adjacent water wave crests is 2π and (53) becomes

$$2d = 2\lambda_w \cos^2\theta \tag{54}$$

or

$$\lambda = 2\lambda_w \cos^2\theta \tag{55}$$

which is identical with (34).

3. DISCUSSION

An analysis for scattering of scalar waves in air by water waves is developed. In the analysis we have made the assumption that the wake extends to infinity. In cases where noise or clutter is present the wake would attenuate and would not continue indefinitely behind the pressure point. The effect of clutter or dissipation may be introduced into this analysis; however, the condition for resonance given in (55) remains unaltered. The dissipation factor that is introduced into the hydrodynamical equations (21)-(23) eliminates the standing wave that extend an infinite distance in back of the pressure point. By considering the scattered field as a generalized function (Hufford, 1972) and taking the limit as the dissipation tends to zero we obtain the result given in (55).

The result in (55) is interesting from another viewpoint. Kurss and Crombie (1971) derived the following resonance condition

$$\lambda = 2\lambda_w \cos(\theta - \alpha c) \qquad (56)$$

Their analysis was based on examining the behavior of the scattered field only in the vicinity of the cuspidal lines. From figure 2 we see that the cuspidal region is only a small portion of the wake. An excellent photograph showing the intensity of the wake near the track of the pressure point is given in Feynman et al. (1963). This photograph shows the intensity near the track to be nearly as great as the intensity near the cuspidal lines. Numerically generated wakes (Tuck et al., 1971) also show a strong component of wake along the track. Thus, it is not surprising that the resonance relation based on the entire wake is different from the resonance condition based upon the cuspidal component of the wake.

4. REFERENCES

Feynman, R. P., Leighton, R. B. and Sands, M. (1963), The Feynman Lectures on Physics, Vol. 1, Addison-Wesley Publishing Co. (New York), p. 51-8.

Havelock, T. H. (1918), Wave resistance: some cases of three-dimensional fluid motion, Proc. of the Royal Society, A., Vol. 95, Nov. 27, p. 147, equations (1) and (2).

Hogner, E. (1923), A contribution to the theory of ship waves, Ark. Mat. Astr. Fys., 17, No. 12, pp. 1-50.

Hufford, G. A. (1972), Scattering by a perturbed plane and generalized functions, unpublished notes.

Lord Kelvin (1887), On ship waves, Proc. of the Royal Scoeity, Vol. 42, p. 80.

Kurss, H., and Crombie, D. D. (1971), private communication.

Sir Horace Lamb (1879), Hydrodynamics, Dover Publications, pp. 384-440.

Peters, A. S. (1949), A new treatment of the ship wave problem, Comm. Pure Appl. Math., Vol. 2, pp. 123-148.

Sommerfeld, A. (1950), Mechanics of deformable bodies, Academic Press (New York), Ch. 5.

Tuck, E. O., Collins, J. I. and Wells, W. H. (1971), On ship wave patterns and their spectra, Jour. of Ship Research, Vol. 15, No. 1, pp. 11-21.

Ursell, F. (1960), On Kelvin's ship-wave pattern, Jour. of Fluid Mech., Vol. 8, pp. 418-431.

5. ACKNOWLEDGEMENT

The authors wish to thank Dr. Donald E. Barrick, Dr. James R. Wait and Mr. D. D. Crombie for their many constructive suggestions during this investigation. The authors also thank Mr. Alvin Van Every for suggesting the problem.

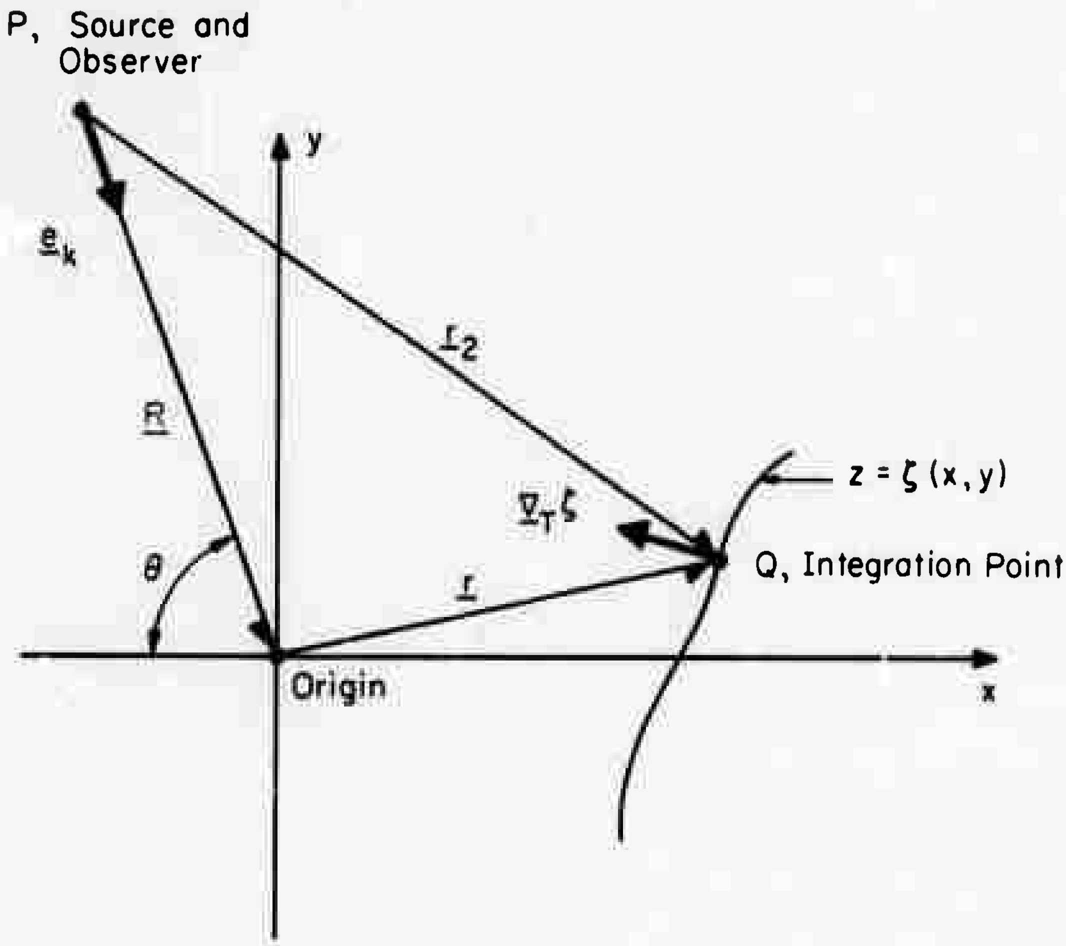

Figure 1. Geometry for scattering problem.

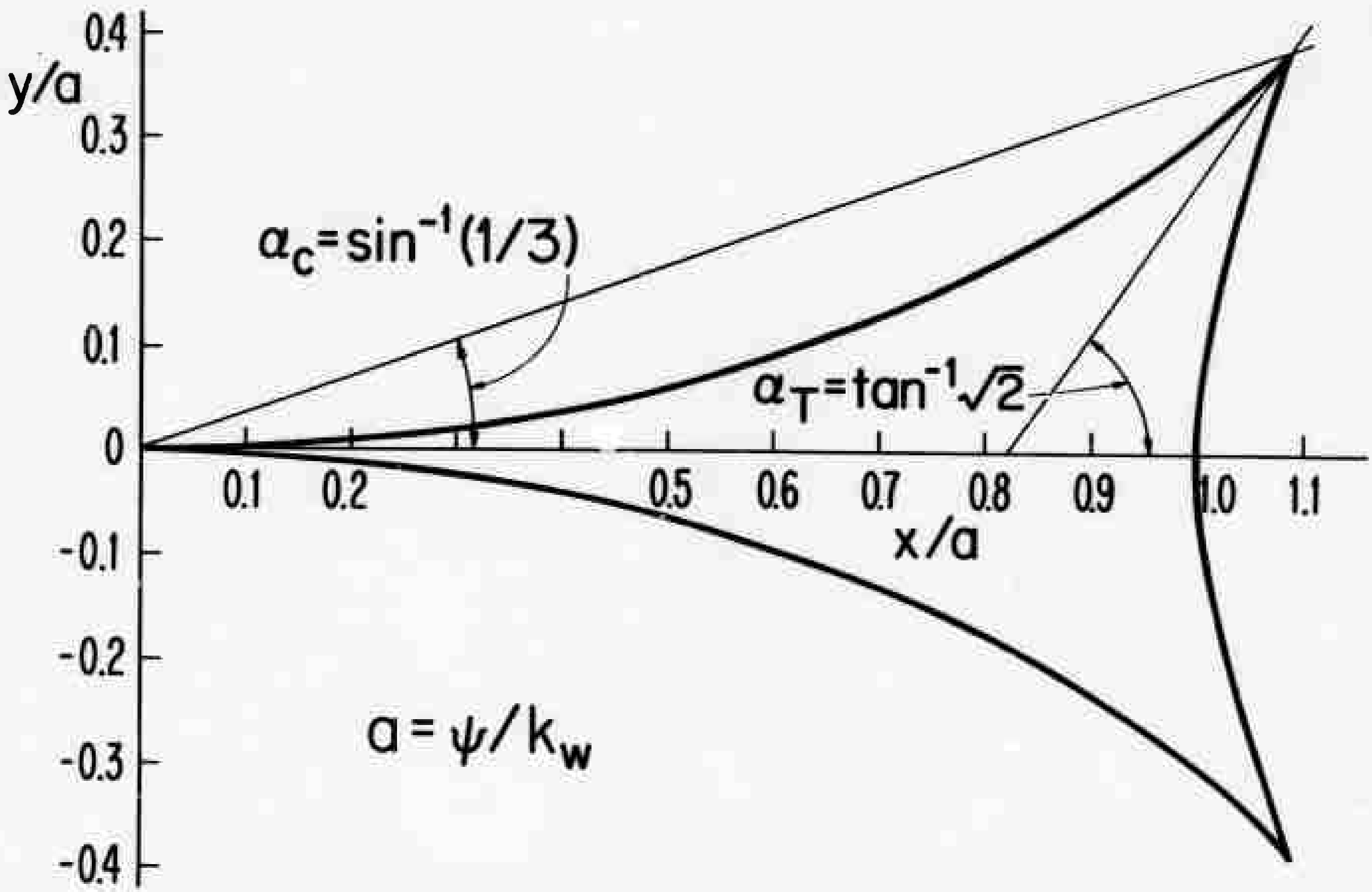

Figure 2. A curve of constant water wave phase showing the lengthwise and transverse components.

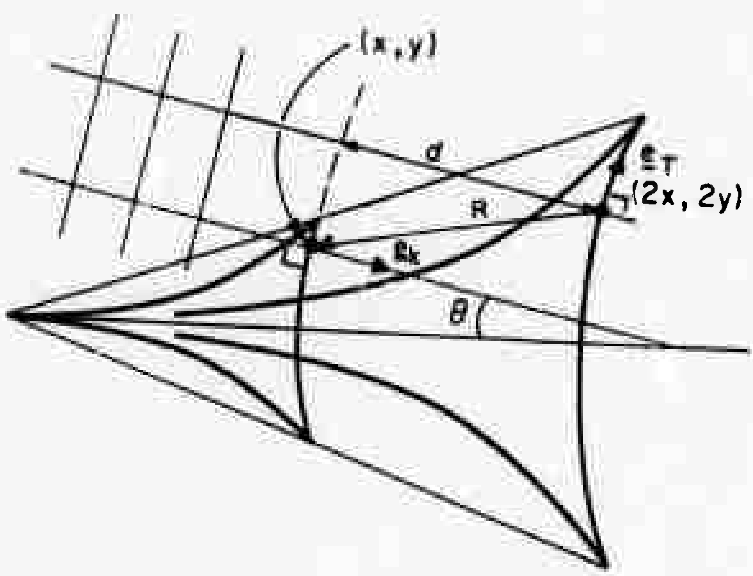

Figure 3. A geometrical interpretation of the resonance condition.